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Baroclinic instability of large-amplitude geostrophic flows

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This paper examines the large-scale dynamics of a layer of stratified fluid on the β-plane. A three-dimensional asymptotic system is derived which governs geostrophic flows with *large* displacement of isopycnal surfaces. This is then reduced to a two-dimensional set of equations which describe the interaction of a baroclinic 'quasi-mode' with *arbitrary* vertical profile and barotrophic motion. The baroclinic instability of large-amplitude zonal flows with vertical shear is studied within the framework of these equations. In the case where the displacement of isopycnal surfaces is small, the results obtained should overlap with the 'traditional' baroclinic instability of quasi-geostrophic (small-amplitude) flows. In order to compare the two types of instability, the quasi-geostrophic boundary-value problem is solved asymptotically for the case of long-wave disturbances and weak β-effect (the latter limit of quasi-geostrophic theory has not been considered previously). The instability that is found is linked to the Hamiltonian structure of the governing equations. The equations derived are generalized for the case of more than one baroclinic quasi-mode.

1. Introduction

The dynamics of large-scale density-driven flows in the ocean are governed by three non-dimensional parameters:

(i) The Rossby number ϵ which characterizes balance of nonlinear effects and rotation:

$$\epsilon = U/Lf,$$

where U is the effective velocity scale, L is the horizontal spatial scale of the motion and f is the Coriolis parameter. All large-scale currents in the ocean (except, possibly, the Gulf Stream) are geostrophic: $\epsilon \ll 1$.

(ii) The β-effect characterized by

$$\beta = (\cot\lambda)\, L_d/R,$$

where $L_d = (gH\,\delta\rho/\rho_0)^{\frac{1}{2}} f^{-1}$ is the internal deformation radius, g is the acceleration due to gravity, H is the total depth of the ocean, $\delta\rho/\rho_0$ is the relative horizontal density variation, R is the Earth's radius and λ is the latitude. Since L_d is always much smaller than R, it follows that β is small.

(iii) The parameter $\delta h/h$, namely the relative displacement of isopycnal surfaces (δh refers to variations over the scale L). The Rossby waves are characterized by $\delta h/h \ll 1$, whereas the isopycnal surfaces of oceanic fronts usually outcrop onto the surface of the ocean ($\delta h = h$, cf. figure 1). This paper examines currents with $\delta h/h \sim 1$ (large-amplitude flows).

Traditionally, density-driven currents were examined using the two-layer model of stratification. There are three approaches to two-layer flows.

The first approach (e.g. Killworth, Paldor & Stone 1984; Paldor & Killworth 1987; Paldor & Ghil 1990) is based on the direct analysis of the exact equations of two-layer fluid dynamics. Unfortunately, these equations are very complex and in most cases cannot be solved analytically.

The second approach employs the assumption that the depth of the upper layer is much smaller than the total depth of the fluid (e.g. Griffiths, Killworth & Stone 1982; Killworth 1983; Cushman-Roisin 1986). Direct comparison of the one-layer and two-layer results (Killworth 1983) shows that the one-layer model is adequate only if this parameter is less than 0.01. Unfortunately, in the real ocean this parameter is not very small: $h/H \sim \frac{1}{3}-\frac{1}{5}$.

The third approach to the dynamics of two-layer density-driven flows is based on the assumption

$$\epsilon \ll 1$$

(Benilov 1992). It was demonstrated that large-amplitude geostrophic dynamics depend strongly on the ratio of small parameters ϵ and β: two systems of equations were derived for the cases of weak and strong β-effect ($\beta \sim \epsilon^{\frac{3}{2}}$ and $\beta \sim \epsilon$, respectively).

The present paper examines the case

$$\beta \sim \epsilon^{\frac{3}{2}} \ll 1, \quad \delta h/h \sim 1$$

within the framework of a *continuous* model of stratification. An asymptotic system of three-dimensional equations is derived (§2) and reduced to a two-dimensional set, which describes interaction of a baroclinic 'quasi-mode' (with arbitrary vertical profile) and barotropic motion (§3). Although this reduction imposes a severe constraint on allowable initial conditions, the two-dimensional set can describe the baroclinic instability of zonal flows with an arbitrary vertical shear (§4). In order to compare this instability to the 'traditional' baroclinic instability of small-amplitude quasi-geostrophic flows, the latter is examined asymptotically using the (long-wave + weak β-effect) approximation (§5). The instability of large-amplitude flows is linked to the Hamiltonian structure of the governing equations in §6. A system which describes the barotropic motion and more than one baroclinic quasi-modes is derived in §7.

2. Basic equations

The equations governing a layer of ideal stratified fluid on the β-plane are

$$u_t + uu_x + vu_y + wu_z + P_x = (1+\beta y)\, v, \tag{1a}$$

$$v_t + uv_x + vv_y + wv_z + P_y = -(1+\beta y)\, u, \tag{1b}$$

$$P_z = -\rho, \tag{1c}$$

$$\rho_t + u\rho_x + v\rho_y + w\rho_z = 0, \tag{1d}$$

$$u_x + v_y + w_z = 0. \tag{1e}$$

Here

$$\left.\begin{aligned} x = \frac{\tilde{x}}{L_d}, \quad y = \frac{\tilde{y}}{L_d}, \quad z = \frac{\tilde{z}}{H}, \quad t = \tilde{t}f, \quad u = \frac{\tilde{u}}{L_d f}, \quad v = \frac{\tilde{v}}{L_d f}, \quad w = \frac{\tilde{w}}{Hf}, \\ \rho = \frac{\tilde{\rho} - \rho_0}{\delta\rho}, \quad P = \frac{\tilde{P} - gz}{gH\,\delta\rho/\rho_0}, \end{aligned}\right\} \tag{2}$$

where the dimensional variables (the spatial coordinates (x, y, z); the time t; the fluid velocity (u, v, w); the pressure P and the density ρ) are marked with tildas.

The no-flow condition at the rigid boundaries are

$$w = 0 \quad \text{at} \quad z = -1, 0. \tag{3}$$

Instead of the continuity equation (1 *e*), we shall derive the vorticity equation according to the following 'recipe':

$$(1b)_x - (1a)_y - (1-\beta y)(1e).$$

Following routine calculations we obtain

$$(v_x - u_y)_t + u(v_x - u_y)_x + v(v_x - u_y)_y + (u_x + v_y)(v_x - u_y) + [w(v_x - u_y)_z + (w_x v_z - w_y u_z)] = (1+\beta y) w_z - \beta v. \tag{4}$$

Since (1 *e*) was included in derivation of (4), system (1 *a–d*), (4) is equivalent to the original set (1 *a–e*).

We are concerned with large-amplitude flows where

(i) the horizontal and vertical variations of density are of the same order;

(ii) the displacement of isopycnal surfaces is of the order of the total depth of the layer.

Accordingly, ρ, P and z should be scaled by unity:

$$\rho = \rho', \quad P = P'; \qquad z = z'. \tag{5a–c}$$

Within the framework of the geostrophic approximation the scaling factors should satisfy the following conditions

$$\frac{\text{scale of } P}{\text{scale of } x \text{ and } y} = \text{scale of } u \text{ and } v,$$

$$\frac{\text{scale of } u \text{ and } v}{\text{scale of } x \text{ and } y} = \frac{1}{\text{scale of } t} = \epsilon,$$

$$\epsilon \frac{\text{scale of } u \text{ and } v}{\text{scale of } x \text{ and } y} = \frac{\text{scale of } w}{\text{scale of } z};$$

where ϵ is the Rossby number. From a physical viewpoint, these conditions mean that the horizontal spatial scale is much bigger than L_d and the vertical density advection is much weaker than the horizontal advection. Accordingly, we have

$$x = \epsilon^{-\frac{1}{2}} x', \quad y = \epsilon^{-\frac{1}{2}} y'; \qquad t = \epsilon^{-1} t'; \tag{5d–f}$$

$$u = \epsilon^{\frac{1}{2}} u', \quad v = \epsilon^{\frac{1}{2}} v', \quad w = \epsilon^2 w'. \tag{5g}$$

We also assume that

$$\beta = \epsilon^{\frac{3}{2}} \beta', \tag{5h}$$

which means that the β-effect terms in (1 *a, b*) are of the order of the nonlinear ageostrophic terms.

Substitution of (5) into (1 *a–d*), (4) yields (primes omitted):

$$\epsilon(u_t + uu_x + vu_y) + \epsilon^2 w u_z + P_x = (1+\epsilon\beta y) v, \tag{6a}$$

$$\epsilon(v_t + uv_x + vv_y) + \epsilon^2 w v_z + P_y = -(1+\epsilon\beta y) u, \tag{6b}$$

$$P_z = -\rho, \tag{6c}$$

$$\rho_t + u\rho_x + v\rho_y + \epsilon w \rho_z = 0, \tag{6d}$$

$$(v_x - u_y)_t + u(v_x - u_y)_x + v(v_x - u_y)_y + (u_x + v_y)(v_x - u_y) + \epsilon[w(v_x - u_y)_z + (w_x v_z - w_y u_z)] = (1+\epsilon\beta y) w_z - \beta v. \tag{6e}$$

Omitting small terms from (6*a–c*), we express u, v and ρ via P:

$$v = P_x, \quad u = -P_y, \quad \rho = -P_z. \tag{7}$$

Substituting (7) into (6*d*, *e*) and dropping small terms, we obtain

$$w_z = \nabla^2 P_t + \mathrm{J}(P, \nabla^2 P) + \beta P_x; \tag{8}$$

$$P_{zt} + \mathrm{J}(P, P_z) = 0, \tag{9a}$$

where $\mathrm{J}(P, Q) = P_x Q_y - P_y Q_x$ is the Jacobian operator. Integrating (8) with respect to z and taking into account boundary conditions (3), we obtain

$$\int_{-1}^{0} [\nabla^2 P_t + \mathrm{J}(P, \nabla^2 P) + \beta P_x]\,\mathrm{d}z = 0. \tag{9b}$$

Equations (9*a*) and (9*b*) form a closed system governing $P(x, y, z, t)$.

3. Two-dimensional reductions of system (9)

It is to be expected that for the two-layer stratification, the three-dimensional integro-differential system (9) can be reduced to the system derived directly from the two-layer shallow-water equations by Benilov (1992). It appears, however, that in some cases of *continuous* stratification, (9) can also be reduced to a two-dimensional differential system.

3.1. *Density-driven currents*

System (9) is compatible with the following ansatz:

$$P(x, y, z, t) = \bar{P}(z) + \Psi(x, y, t) + \Phi(x, y, t)\,\phi(z), \tag{10}$$

where $\bar{P}$ describes the background field of stratification, Ψ and Φ are, respectively, the amplitudes of barotropic and baroclinic components of the motion. Since vertical profiles of the two components are steady in time, they will be referred to as 'quasi-modes'. Without loss of generality $\phi(z)$ can be assumed to satisfy the following conditions:

$$\int_{-1}^{0} \phi\,\mathrm{d}z = 0, \quad \int_{-1}^{0} \phi^2\,\mathrm{d}z = 1 \tag{11a, b}$$

(the pattern of isopycnal surfaces, corresponding to ansatz (10) with

$$\Phi \to 0, \quad \Psi \to 0 \quad \text{as} \quad x \to -\infty;$$

$$\Phi \to \mathrm{const}_1 \neq 0, \quad \Psi \to \mathrm{const}_2 \neq 0 \quad \text{as} \quad x \to +\infty$$

is shown on figure 1*a*). Substituting (10) into (9) and taking into account (11), we obtain the following system of equations which govern Ψ and Φ:

$$\Phi_t + \mathrm{J}(\Psi, \Phi) = 0, \quad \nabla^2 \Psi_t + \mathrm{J}(\Psi, \nabla^2 \Psi) + \mathrm{J}(\Phi, \nabla^2 \Phi) + \beta \Psi_x = 0. \tag{12}$$

It is noted that the resulting equations do not depend on the background stratification field $\bar{P}(z)$, which indicates that large-amplitude flows are sensitive only to horizontal gradients of density.

3.2. *Fronts*

It is remarked that the above pattern of isopycnal surfaces (figure 1*a*) does not describe the important feature of oceanic fronts wherein isopycnal surfaces converge to the surface in a 'bunch' (figure 1*b*). The ocean in this case can be subdivided into two

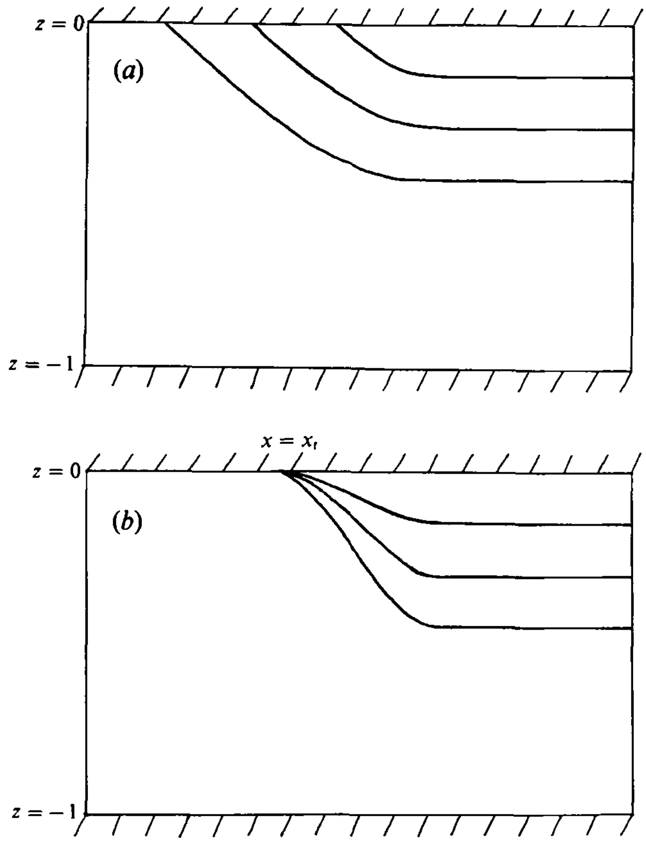

FIGURE 1. Distribution of isopycnal surfaces in (*a*) a density-driven current (ansatz (10)); (*b*) an oceanic front (ansatz (13)).

continuously stratified layers: the bottom layer with steady stratification $\bar{P}(z)$ and the upper layer with a horizontally inhomogeneous stratification field. The upper layer is 'squeezed' or 'stretched' by its depth $h(x, y, t)$, which vanishes at the outcrop of the front: $h(x_f, y, t) = 0$. This pattern corresponds to the following ansatz:

$$P(x, y, z, t) = \bar{P}(z) + F(x, y, t) + G(x, y, t)\, \psi[z/h(x, y, t)], \tag{13}$$

$$\psi|_{z \leqslant -1} = 0. \tag{14a}$$

where $\psi(z)$ describes the stratification of the upper layer. Without loss of generality, ψ can be assumed to satisfy

$$\int_{-1}^{0} \psi \,\mathrm{d}z = 1. \tag{14b}$$

The density variation ρ has its minimum value at the surface, where it can be assumed constant†:

$$\rho|_{z=0} = -P_z|_{z=0} = -1.$$

Substitution of (13) into the last equality yields

$$G = h(x, y, t)/\psi'(0),$$

† This corresponds to an infinitesmimal layer with fast-varying density profile at $x \leqslant x_f$ (cf. figure 1*b*).

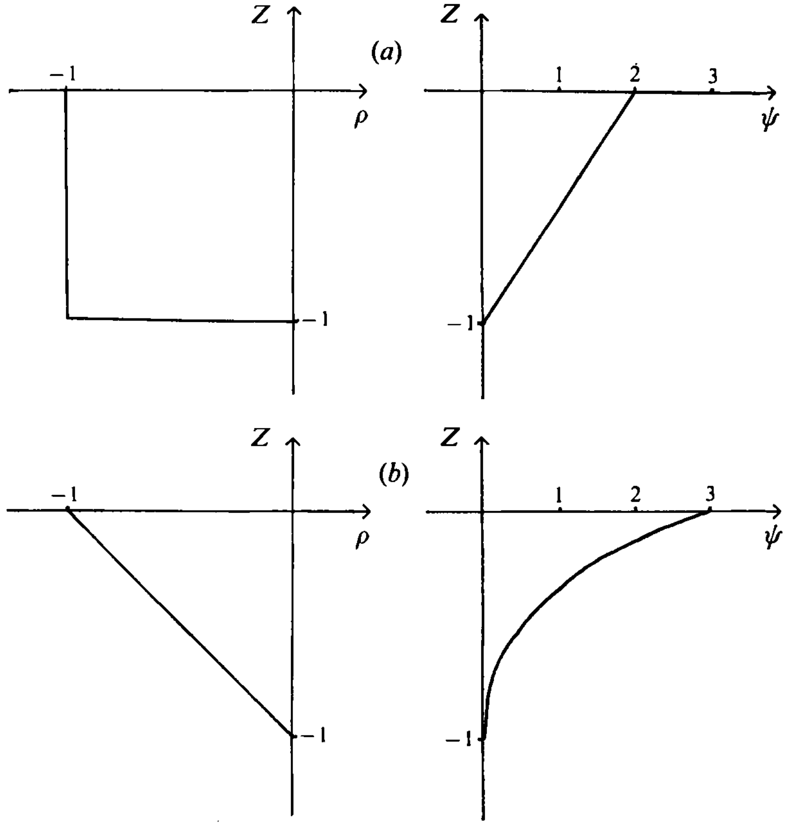

FIGURE 2. Two models of stratification for oceanic fronts. ψ describes vertical structure of the pressure field, ρ is the corresponding non-dimensional density variation: (*a*) two-layer model; (*b*) model with linear stratification of the upper layer.

where the prime denotes differentiation with respect to z. Introducing the amplitude of the barotropic component:

$$\Psi(x,y,t) = \int_{-1}^{0} [P(x,y,z,t) - \bar{P}(z)]\,\mathrm{d}z = F + G\int_{-1}^{0} \psi\frac{z}{h}\,\mathrm{d}z = F + \frac{h^2}{\psi'(0)},$$

ansatz (13) can be rewritten in the form

$$P(x,y,z,t) = \bar{P}(z) + \Psi(x,y,t) + h(x,y,t)\,\frac{\psi[z/h(x,y,t)] - h(x,y,t)}{\psi'(0)}. \tag{15}$$

Substituting (15) into (9), we obtain

$$\left.\begin{aligned} h_t + \mathrm{J}(\Psi, h) &= 0, \\ \nabla^2\Psi_t + \mathrm{J}(\Psi, \nabla^2\Psi) + \alpha^2\nabla\cdot\mathrm{J}[h, h(\gamma - h)\nabla h] + \beta\Psi_x &= 0 \end{aligned}\right\} \tag{16}$$

where

$$\alpha = \frac{2}{\psi'(0)}, \quad \gamma = \tfrac{1}{4}\int_{-1}^{0} (2\psi^2 + z^2\psi'^2)\,\mathrm{d}z \tag{17}$$

are constants which depend on the stratification profile of the upper layer. For example, the two-layer model (figure 2*a*) corresponds to

$$\psi = 2(z+1)\,\theta(z+1), \quad \alpha = 1, \quad \gamma = 1;$$

where $\theta(z)$ is the Heaviside step function, and system (14) coincides with the two-layer equations derived by Benilov (1992). One can also consider a model with linear stratification of the upper layer (figure 2*b*):

$$\psi = 3(z+1)^2\theta(z+1), \quad \alpha = \tfrac{1}{3}, \quad \gamma = \tfrac{6}{5}.$$

Further models of stratification will not be considered here. Instead, we shall demonstrate that system (16) is equivalent to the simpler system (12) regardless of the stratification model adopted.

Evidently, the change of variables

$$h \to \Phi(h) = \alpha \int_0^h [h'(\gamma - h')]^{\frac{1}{2}}\,\mathrm{d}h' \tag{18}$$

reduces (16) to (12). Note, however, that transformation (18) exists only if the integrand $[h'(\gamma-h')]^{\frac{1}{2}}$ is real in the interval $(0 \leqslant h' \leqslant 1)$,† that is when

$$\gamma \geqslant 1. \tag{19}$$

where γ is defined by (17) and (14). Proof of this important inequality is given in the Appendix.

Thus, system (12) describes both large-amplitude density-driven currents and fronts. Vertical structure of the flow within its framework is 'parameterized' – this represents an important advantage over the exact system (1) or asymptotic equations (9).

4. Baroclinic instability of zonal flows

It is clear that both substitutions (10) or (15) impose severe constraints on the initial conditions allowed for system (12). Ansatz (10), for example, shows that system (12) is valid only if the vertical and horizontal variables in the initial distribution of pressure can be separated:

$$P(x,y,z) = \bar{P}(z) + \Psi(x,y) + \Phi(x,y)\,\phi(z). \tag{20}$$

Nevertheless, system (12), in contrast with any two-layer system, does describe at least some instances of flows with non-trivial three-dimensional structure, the most important being the baroclinic instability of a zonally homogeneous current. At the same time, consideration of baroclinic instability within the framework of system (12) should not be difficult: since (12) does not explicitly depend on the vertical structure of the current, the corresponding boundary-value problem will contain only *constant* coefficients.

4.1. *Growth rate of the instability*

Consider the following steady solution to (12):

$$\Psi_0 = -Uy, \quad \Phi_0 = -Vy; \tag{21}$$

where U and V are constants. Solution (21) describes a zonally homogeneous current $\bar{u}(z)$ with an arbitrary vertical profile:

$$\bar{u}(z) = U + V\phi(z). \tag{22}$$

Linearizing (12) against the background of solution (21) according to

$$\Psi(x,y,t) = -Uy + \psi(x,y,t), \quad \Phi(x,y,t) = -Vy + \phi(x,y,t);$$

† Since h represents the dimensional thickness of the upper layer scaled by the total depth of the fluid, it must be positive and smaller than unity.

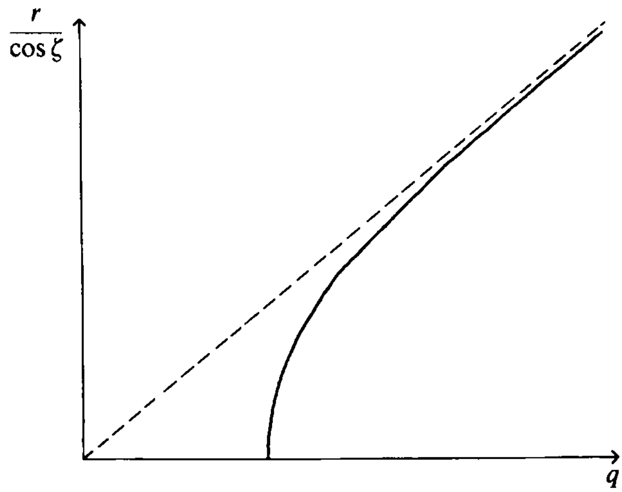

FIGURE 3. Growth rate r versus the wavenumber $q = (k^2+l^2)^{\frac{1}{2}}$ of the disturbances. ζ is the angle between the flow and the wavenumber.

we obtain $$\phi_t + U\phi_x - V\psi_x = 0, \quad \nabla^2\psi_t + U\nabla^2\psi_x + V\nabla^2\phi_x + \beta\psi_x = 0.$$

The harmonic-wave solution to this system is

$$\psi = \text{const}_1 \exp(\mathrm{i}\omega t - \mathrm{i}kx - \mathrm{i}ly), \quad \phi = \text{const}_2 \exp(\mathrm{i}\omega t - \mathrm{i}kx - \mathrm{i}ly);$$

where $$\omega = kU + k\frac{-\beta \pm [\beta^2 - 4V^2(k^2+l^2)^2]^{\frac{1}{2}}}{2(k^2+l^2)}. \tag{23a}$$

Evidently, if $$k^2 + l^2 > \beta/2V,$$

the frequency ω is complex, which indicates instability with the growth rate

$$r = \frac{k}{2(k^2+l^2)} \operatorname{Im}[\beta^2 - 4V^2(k^2+l^2)^2]^{\frac{1}{2}}$$

(cf. figure 3). Thus, all zonal flows described by system (12) are unstable.

It is worth noting that the dispersion relation of unstable disturbances depends only on integral (mean) characteristics of the flow:

$$V^2 = \int_{-1}^{0} [\bar{u}(z) - U]^2 \,\mathrm{d}z, \quad U = \int_{-1}^{0} \bar{u}(z)\,\mathrm{d}z \tag{23b}$$

(these equalities were obtained using (22) and (11)). The growth rate r does not even depend on the barotropic component U, but is determined by the baroclinic component and β-effect. The latter is a stabilizing factor – thus, the most unstable disturbances are short (the β-effect mainly affects long waves).

It should be emphasized that (23a) is valid only if the dimensional wavelength of the perturbation is smaller than the deformation radius (this condition follows from scaling (5)). In terms of the non-dimensional scaled variables this means that

$$k^2 + p^2 \ll \epsilon^{-1}.$$

Nevertheless, one can hope that even with the violation of this condition, (23) provides a qualitatively correct estimate of the growth rate of the instability. Since

	(*a*)	(*b*)
timescale	$\gtrsim \epsilon^{-1}/f$	$\gtrsim \epsilon^{-1}/f$
vertical spatial scale	$\sim H$	$\sim H$
horizontal spatial scale	$\gtrsim \epsilon^{-\frac{1}{2}} L_d$	$\gtrsim L_d$
displacement of isopycnal surfaces	$\sim H$	$\lesssim \epsilon H$
velocity scale	$\lesssim \epsilon^{\frac{1}{2}} fL_d$	$\lesssim \epsilon fL_d$
slope of isopycnal surfaces	$\lesssim \epsilon^{\frac{1}{2}} H/L_d$	$\lesssim \epsilon H/L_d$
β	$\lesssim \epsilon^{\frac{3}{2}}$	$\lesssim \epsilon$

TABLE 1. Comparison between the dimensional parameters of flows described by (*a*) the present theory and (*b*) the quasi-geostrophic equation governing small-amplitude currents. Here f is the Coriolis parameter, L_d is the deformation radius, H is the total depth of the ocean and ϵ is the Rossby number.

disturbances with shorter wavelengths are of internal-wave (rather than planetary-wave) nature, they are likely to be stable; thus, the maximum growth takes place at wavelengths of the order of the deformation radius:

$$\left.\begin{aligned} \tilde{r} \sim \tilde{V}/L_d \quad \text{as} \quad \tilde{k} \sim 1/L_d, \\ \tilde{V}^2 = \int_{-H}^{0} [\tilde{u}(z) - \tilde{U}]^2 \, dz, \quad \tilde{U} = \int_{-H}^{0} \tilde{u}(z) \, dz; \end{aligned}\right\} \tag{24}$$

where the dimensional variables are marked with tildas.

System (12) has another steady solution:

$$\Psi \equiv 0, \quad \Phi = -V(y \cos \zeta - x \sin \zeta); \tag{25}$$

which describes a non-zonal current in the upper layer and a countercurrent in the bottom layer, such that the total mass flux is equal to zero. Although solution (25) is not steady within the framework of the original equations (1), the timescale of its evolution is abnormally long ($\sim \epsilon^{-2}/f$) and the asymptotic system (12) 'treats' it as steady.

With regards to stability, (25) is similar to the case of zonal flows. The dispersion relation of linear perturbations,

$$\omega = k \frac{-\beta \pm \{\beta^2 - 4V^2(k^2+l^2)^2[(k \cos \zeta + l \sin \zeta)/k]^2\}^{\frac{1}{2}}}{2(k^2+l^2)},$$

demonstrates that all non-zonal currents are unstable.

4.2. *Discussion*

In order to clarify the question of correspondence of the above instability and the 'traditional' baroclinic instability of small-amplitude quasi-geostrophic currents, we compare in table 1 the dimensional parameters of flows described by: (*a*) the present theory; (*b*) the quasi-geostrophic equation governing small-amplitude currents (cf. Kamenkovich & Reznik 1978; Pedlosky 1987; and references therein). The parameters in the table are derived from the scaling equalities (2), (5) and Pedlosky (1987, Chap. 6).

Table 1 indicates that the small-amplitude limit of (*a*) should coincide with the (long-wave + weak β-effect) limit of (*b*). The latter limit of the quasi-geostrophic equation seems not to have been previously considered in the literature. It will be examined in the next section.

5. Long-wave quasi-geostrophic instability with weak β-effect

In terms of the non-dimensional scaled variables used above, the quasi-geostrophic equation is (e.g. Pedlosky 1987)

$$[\epsilon\Delta P+(nP_z)_z]_t+\mathrm{J}[P,\epsilon\Delta P+(nP_z)_z]+\epsilon\beta P_x=0, \tag{26a}$$

where $n(z)=f^2/N^2$ is the stratification field and $N(z)$ is the Väsälä frquency. Equation (26*a*) should be supplemented by the boundary condition

$$n[P_{zt}+\mathrm{J}(P,P_z)]=0 \quad \text{at} \quad z=-1,0. \tag{26b}$$

Linearizing (26) against the background of a steady shear flow:

$$P(x,y,z,t)=-y\bar{u}(z)+p(z)\exp(\mathrm{i}\omega t-\mathrm{i}kx-\mathrm{i}ly),$$

we obtain
$$(\omega-k\bar{u})[\epsilon(k^2+l^2)p-(np_z)_z]+\epsilon\beta kp-k(n\bar{u}_z)_z p=0,$$
$$n[(\omega-k\bar{u})p_z+k\bar{u}_z p]=0 \quad \text{at} \quad z=-1,0.$$

We solve this boundary-value problem using the perturbation expansion

$$p=p_0+\epsilon p_1+\dots, \quad \omega=\omega_0+\epsilon\omega_1+\dots.$$

The solution to the zeroth-order boundary-value problem

$$(\omega_0-k\bar{u})(np_{0z})_z+k(n\bar{u}_z)_z p_0=0,$$
$$n[(\omega_0-k\bar{u})p_{0z}+k\bar{u}_z p_0]=0 \quad \text{at} \quad z=-1,0$$

is
$$p_0=\omega_0-k\bar{u}.$$

The first-order boundary-value problem is

$$(\omega_0-k\bar{u})(np_{1z})_z+k(n\bar{u}_z)_z p_1=(\omega_0-k\bar{u})^2(k^2+l^2)+\beta k(\omega_0-k\bar{u})-\omega_1[n(\omega_0-k\bar{u})_z]_z, \tag{27a}$$

$$n[(\omega_0-k\bar{u})p_{1z}+k\bar{u}_z p_1]=n\omega_1(\omega_0-k\bar{u})_z \quad \text{at} \quad z=-1,0. \tag{27b}$$

Integrating (27*a*) from $z=-1$ to $z=0$ and taking into account (27*b*), we obtain the following equation for ω_0:

$$(k^2+l^2)\int_{-1}^{0}(\omega_0-k\bar{u})^2\,\mathrm{d}z+\beta k\int_{-1}^{0}(\omega_0-k\bar{u})\,\mathrm{d}z=0.$$

Evidently, the solution to this equation coincides with (23). It should be emphasized that, although the two expressions coincide exactly, the dispersion relation (23) does not imply an assumption of small displacement of isopycnal surfaces: scaling (5) proves that the displacement of isopycnal surfaces was assumed large.

6. Hamiltonian structure of system (12)

In order to clarify the mathematical background of the instabilities found, (12) will be considered as a Hamiltonian system.

System (12) conserves the following invariant of motion:

$$\mathscr{F}=\iint_{-\infty}^{+\infty}F(\Phi)\,\mathrm{d}x\,\mathrm{d}y,$$

where $F(\Phi)$ is an arbitrary function. In terms of exact equations (6), the above invariant corresponds to

$$\mathscr{G} = \iint_{-\infty}^{+\infty} \int_{-1}^{0} G(\rho-\bar{\rho})\, \mathrm{d}z\, \mathrm{d}x\, \mathrm{d}y, \quad \bar{\rho}(z) = -\bar{p}_z(z);$$

where $G(\rho)$ is also an arbitrary function (not necessarily equal to F). If $G(\rho) = \rho$ and $F(\Phi) = \Phi$, both invariants represent mass conservation law.

It is worth noting that the energy invariant of system (6)

$$\mathscr{E} = \iint_{-\infty}^{+\infty} \int_{-1}^{0} [(\rho-\bar{\rho})\, z + \tfrac{1}{2}\epsilon(u^2+v^2)]\, \mathrm{d}z\, \mathrm{d}x\, \mathrm{d}y \tag{28}$$

'survives' the transition to the asymptotic system (12) only as the integral of potential energy. Indeed, substituting (10) into (28) and taking into account (11*a*), we have

$$\lim_{\epsilon\to 0} \mathscr{E} = \iint_{-\infty}^{+\infty} \Phi\, \mathrm{d}x\, \mathrm{d}y \int_{-1}^{0} \phi' z\, \mathrm{d}z = -\phi'(-1) \iint_{-\infty}^{+\infty} \Phi\, \mathrm{d}x\, \mathrm{d}y,$$

which also shows that the potential energy of geostrophic motion is proportional to the mass invariant. Clearly, the kinetic energy

$$\mathscr{K} = \tfrac{1}{2} \iint_{-\infty}^{+\infty} [(\nabla\Psi)^2 + (\nabla\Phi)^2]\, \mathrm{d}x\, \mathrm{d}y,$$

(obtained by substituting equalities (7*a*) and (10) into the second term of the energy integral (28)) cannot be conserved separately. This can also be verified via straightforward calculation of $\mathrm{d}\mathscr{K}/\mathrm{d}t$.

System (12) also conserves an invariant of motion having no analogue in terms of the exact equations (6) and differing from the kinetic energy in sign:

$$\mathscr{H} = \tfrac{1}{2} \iint_{-\infty}^{+\infty} [(\nabla\Psi)^2 - (\nabla\Phi)^2]\, \mathrm{d}x\, \mathrm{d}y. \tag{29}$$

Although $\mathscr{H}$ has no obvious physical meaning,† it plays the role of the Hamiltonian of system (12). In order to demonstrate this, we define the Poisson brackets:

$$\{\mathscr{P}, \mathscr{Q}\} = \iint_{-\infty}^{+\infty} \left[\Omega \mathrm{J}\left(\frac{\delta\mathscr{P}}{\delta\Omega}, \frac{\delta\mathscr{Q}}{\delta\Omega}\right) + \Phi \mathrm{J}\left(\frac{\delta\mathscr{P}}{\delta\Phi}, \frac{\delta\mathscr{Q}}{\delta\Omega}\right) + \Phi \mathrm{J}\left(\frac{\delta\mathscr{P}}{\delta\Omega}, \frac{\delta\mathscr{Q}}{\delta\Phi}\right)\right] \mathrm{d}x\, \mathrm{d}y,$$

where $\Omega = \nabla^2\Psi + \beta y$ and $\mathscr{P}$, $\mathscr{Q}$ are arbitrary functionals depending on Φ and Ω. Now system (12) can be rewritten in the Hamiltonian form:

$$\Phi_t + \{\mathscr{H}, \Phi\} = 0, \quad \Omega_t + \{\mathscr{H}, \Omega\} = 0.$$

Note that Hamiltonian (29) is of sign-indefinite type.‡

Generally speaking, the fact that the Hamiltonian of a dynamic system may change its sign is usually of profound importance for the stability properties of the system. For example, the ordinary differential equation

$$x_{tt} - x = 0$$

with the Hamiltonian

$$\mathscr{H} = \tfrac{1}{2}[(x_t)^2 - x^2]$$

† One of the referees of this paper suggested that $\mathscr{H}$ may be proportional to the total energy $\mathscr{E}$ minus the particular form of $\mathscr{G}$ that cancels the dominant potential energy.

‡ A very similar, sign-indefinite invariant was mentioned, in a related context, by Cushman-Roisin, Sutyrin & Tang (1992) and Benilov (1992).

describes a particle on the solid parabola turned upside down and evidently has no stable solutions at all. An example of a partial differential system with sign-indefinite Hamiltonian can be seen in

$$\Psi_t - \Phi = 0, \quad \Phi_t + \nabla^2 \Psi = 0; \qquad \mathscr{H} = \tfrac{1}{2} \iint_{-\infty}^{+\infty} [(\nabla\Psi)^2 - \Phi^2] \, dx \, dy.$$

System (30) is of the elliptical type and its only steady solution

$$\Psi = 0, \quad \Phi = 0$$

is unstable with respect to short-wave perturbations.

Although system (12) has a much more complicated structure than the above equations, the link between its sign-indefinite Hamiltonian and short-wave instability seems to be similar.

7. (n + 1)-mode system

System (12) describes the interaction of only two 'quasi-modes' and therefore cannot 'resolve' flows with complicated three-dimensional structure. In order to derive a more general system, we shall consider interaction of one barotropic and n baroclinic quasi-modes:

$$p(x,y,z,t) = \bar{p}(z) + \Psi(x,y,t) + \sum_{j=1}^{n} [\Psi_j(x,y,t) + \Phi_j(x,y,t)\,\phi_j(z)], \quad j = 1,2,\ldots,n. \tag{31}$$

The quasi-modes are defined in n separate layers of equal depths:

$$\phi_j(z) \neq 0 \quad \text{only if} \quad -j/n \leqslant z \leqslant -(j-1)/n,$$

where their profiles satisfy the following conditions:

$$\int_{-(j-1)/n}^{-j/n} \phi_j \, dz = 0, \qquad \int_{-(j-1)/n}^{-j/n} \phi_j^2 \, dz = \frac{1}{n}. \tag{32}$$

In order to render Ψ the barotropic-mode amplitude, we set

$$\sum_{j=1}^{n} \Psi_j = 0, \tag{33a}$$

while the condition of vertical continuity yields

$$\Psi_j + \Phi_j \phi_j(-j/n) = \Psi_{j+1} + \Phi_{j+1} \phi_{j+1}(-j/n), \quad j = 1,2,\ldots,n-1.$$

Without loss of generality we may assume that

$$\phi_j(-j/n) = \phi_j(-(j+1)/n) = 1,$$

which yields

$$\Psi_j + \Phi_j = \Psi_{j+1} + \Phi_{j+1}, \quad j = 1,2,\ldots,n-1. \tag{33b}$$

Finally, substituting ansatz (31) into (9) and taking into account (32) and (33a), we obtain

$$\left.\begin{aligned} &\frac{\partial}{\partial t}\Phi_j + \mathrm{J}(\Psi + \Psi_j, \Phi_j) = 0, \quad j = 1,2,\ldots,n, \\ &\frac{\partial}{\partial t}\nabla^2\Psi + \mathrm{J}(\Psi, \nabla^2\Psi) + \frac{1}{n}\sum_{j=1}^{n} [\mathrm{J}(\Psi_j, \nabla^2\Psi_j) + \mathrm{J}(\Phi_j, \nabla^2\Phi_j)] + \beta\frac{\partial}{\partial x}\Psi = 0. \end{aligned}\right\} \tag{33c}$$

The number of equations in system (33) is equal to the number of unknown functions. Increasing n, we can resolve three-dimensional structure of a given flow with any required accuracy.

Consider the case $n = 2$. Using (33*a, b*) we can express Ψ_1 and Ψ_2 via Φ_1 and Φ_2:

$$\Psi_1 = \tfrac{1}{2}(\Phi_2 - \Phi_1), \quad \Psi_2 = \tfrac{1}{2}(\Phi_1 - \Phi_2).$$

Substituting these expressions into (33*c*), we obtain

$$\frac{\partial}{\partial t}\Phi_1 + \mathrm{J}(\Psi + \tfrac{1}{2}\Phi_2, \Phi_1) = 0, \quad \frac{\partial}{\partial t}\Phi_2 + \mathrm{J}(\Psi + \tfrac{1}{2}\Phi_1, \Phi_2) = 0,$$

$$\frac{\partial}{\partial t}\nabla^2\Psi + \mathrm{J}(\Psi, \nabla^2\Psi) + \tfrac{3}{4}[\mathrm{J}(\Phi_1, \nabla^2\Phi_1) + \mathrm{J}(\Phi_2, \nabla^2\Phi_2)] - \tfrac{1}{4}[\mathrm{J}(\Phi_1, \nabla^2\Phi_2) + \mathrm{J}(\Phi_2, \nabla^2\Phi_1)] + \beta\frac{\partial}{\partial x}\Psi = 0.$$

This system describes one barotropic and two baroclinic quasi-modes, and its structure is quite similar to that of the two-mode system (12).

8. Conclusions

The main result of the present paper is the derivation of the asymptotic system (9), which describes geostrophic flows with large displacement of isopycnal surfaces and weak β-effect.

It was further demonstrated that this three-dimensional system can be reduced to the two-dimensional system (12) which governs the amplitudes of barotropic and baroclinic components of the flow (the amplitudes depend on the horizontal spatial variables and time). It should be emphasized that, in contrast with linear wave systems, the vertical profile of the baroclinic component ('quasi-mode') is arbitrary. System (12) describes both density-driven currents and fronts, which indicates that it can be used as a robust qualitative model of large-amplitude motion in general.

It was demonstrated that within the framework of (12), all large-amplitude zonally homogeneous flows, regardless of their vertical structure, are unstable with respect to short-wave perturbations. The parameters of the instability coincide with the corresponding limit of the quasi-geostrophic baroclinic instability. It was also demonstrated that the Hamiltonian of (12) is sign-indefinite, which indicates that all steady solutions are unstable. This conclusion agrees with the existence of short-wave instability of fronts and currents observed experimentally, and seems to be a strong argument in favour of the present theory.

System (12) describes the interaction of only two quasi-modes, and therefore cannot resolve flows having complicated three-dimensional structure. In this case it can be generalized to describe an arbitrary number n of barclinic quasi-modes – the n-mode system can 'resolve' any given flow with any required accuracy.

Finally, the results obtained can be easily generalized to the case of an uneven bottom, provided that the depth variations are much smaller than the average depth of the ocean.

Appendix. Proof of inequality (19)

Using (17), inequality (19) can be rewritten in the form

$$\tfrac{1}{4}\int_{-1}^{0}(2\psi^2+z^2\psi'^2)\,dz \geqslant 1, \tag{A 1}$$

where $\psi(z)$ is a smooth function, defined on the interval $(-1,0)$ and satisfying constraints (14):

$$\psi(0)=0, \tag{A 2}$$

$$\int_{-1}^{0}\psi\,dz = 1. \tag{A 3}$$

It is convenient to treat γ as a functional depending on $\psi(z)$. Since $\gamma[\psi(z)]$ is positive, it has to have a minimum, where its variation equals to 0. Taking into account constraint (A 3), we have

$$\delta\left[\tfrac{1}{4}\int_{-1}^{0}(2\psi^2+z^2\psi_2)\,dz+\lambda\int_{-1}^{0}\psi\,dz\right]=0 \quad \text{for} \quad \psi=\psi_{\min}; \tag{A 4}$$

where λ is the corresponding Lagrange multiplier. Equation (A 4) can be rewritten in the form of a differential equation

$$\psi_{\min}-\tfrac{1}{2}(z^2\psi'_{\min})'+\lambda=0$$

and easily integrated

$$\psi_{\min}=c_1 z+c_2 z^{-2}-\lambda. \tag{A 5}$$

Substitution of (A 5) into (A 2), (A 3) yields $c_1=2$, $c_2=0$, $\lambda=-2$ and

$$\psi_{\min}=2(z+1), \quad \gamma_{\min}=1.$$

Evidently, the minimum value of γ corresponds to the two-layer model (cf. §3.2).

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